Notes on Relational Incentive Systems

Robert Gibbons, Hongyi Li, **Jin Li, and** Sarah Venables

August 22, 2012

1   Optimal Incentive Contracts

Model

- Output $y \in \{0, 1\}$ where $\Pr[y = 1] = a_1$ - observable but non-contractible.

- Performance Measure $p \in \{0, 1\}$ where $\Pr[p = 1] = a_1 \cos\theta + a_2 \sin\theta$ - contractible.

- Cost of effort $c(a_1, a_2) = \frac{k}{2}(a_1^2 + a_2^2)$

- Agent's outside option: $\bar{u}$

- Principal's outside option: $\bar{\pi} = 0$

1.1   First-Best Effort

First-best effort maximises:
$$\max_{a_1, a_2} \{ a_1 - \frac{k}{2}(a_1^2 + a_2^2) \}$$

so $a_1^{FB} = \frac{1}{k}$, $a_2 = 0$. Total surplus is
$$V_{FB} = \frac{1}{2k}$$

1.2   Optimal Static Contract

Consider the spot game. Since output is non-contractible, in the static game it is not possible to attach any payments to output, so the agent can only be rewarded based on the performance measure. We therefore consider incentive contracts of the form $w = s + bp$ and look for the optimal performance weight b^{SP}.

For a given b, the agent chooses effort to maximise:
$$U_A = \max_{a_1, a_2} \{ s + b(a_1 \sin\theta + a_2 \cos\theta) - \frac{k}{2}(a_1^2 + a_2^2) \}$$

and therefore $a_1(b) = \frac{b}{k}\cos\theta$ and $a_2(b) = \frac{b}{k}\sin\theta$.

The principal then sets b to maximise total surplus:

$$b^{SP} = \text{argmax}\left\{\frac{b}{k}\cos\theta - \frac{k}{2}\left(\left(\frac{b}{k}\cos\theta\right)^2 + \left(\frac{b}{k}\sin\theta\right)^2\right)\right\}$$

So $b^{SP} = \cos\theta$ and the resulting surplus is

$$V^{SP} = \frac{1}{2k}\cos^2\theta$$

1.3 Optimal Relational Contract

In the repeated game, it may be possible for the principal to credibly promise to pay a bonus B based on output. Following the argument in Levin (2003) it is without loss of generality to focus on stationary contracts in which the principal commits to paying a bonus B whenever $y = 1$, the agent supplies a positive amount of effort, and the agent will terminate the relationship if the principal fails to pay the bonus when $y = 1$. For now, focus on the incentives provided by using *only* a relational contract $w = s + By$.

The agent chooses effort to maximise:

$$\max_{a_1,a_2}\{s + Ba_1 - \frac{k}{2}(a_1^2 + a_2^2)\}$$

so $a_1 = \frac{B}{k}$. The first-best requires that $B = 1$. However this promised bonus is only credible if the principal has no incentive to renege; that is, we require:

$$-B + \frac{1}{r}(V_{FB} - \bar{u}) \geq 0$$

Therefore the first-best can be reached whenever

$$r \leq V_{FB} - \bar{u} = \frac{1}{2k} - \bar{u}$$

When the first-best cannot be reached because r is too high (low discount rate) we can look for the second-best relational incentive contract. In this case B^{SB} solves:

$$rB = V(B) - \bar{u} = \frac{1}{2k}(2B - B^2) - \bar{u}$$

1.4 Comparing formal and relational contracts

Figure 1 shows the outcomes of a formal contract and a (second-best) relational contract. Since the outside option is simply the termination of the relationship, the optimal relational contract does not depend on θ. However the success of the formal contract varies with θ - for small θ, the performance measure aligns well with output, so works well. As θ increases, the performance measure puts more weight on a_2, distorting the agent's incentives. Eventually p does so badly that the principal and agent prefer to take the outside option.

Figure 1: Relational vs Formal Contracts
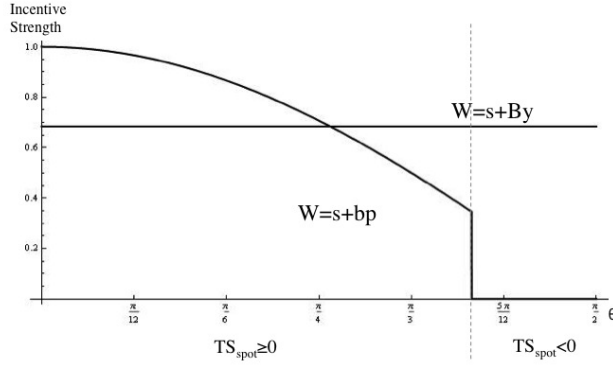
1.5 Combining Formal and Relational Contracts

What if the principal has the option of offering both a formal spot contract and a relational bonus payment: $w = s + bp + By$? In this case the formal payment based on the contractible performance measure may be used to strengthen incentives if the first-best cannot be reached using B. However, the principal and the agent also have the option of reverting to the formal contract should the relationship break down. This may have the effect of making it harder to sustain the relationship, since the cost of termination is reduced.

The agent chooses effort to maximise:

$$\max_{a_1,a_2}\{s + b(a_1 \cos\theta + a_2 \sin\theta) + Ba_1 - \frac{k}{2}(a_1^2 + a_2^2)\}$$

so $a_1 = \frac{1}{k}(b\cos\theta + B)$ and $a_2 = \frac{1}{k}(b\sin\theta)$.

The principal then sets b and B to maximise total surplus:

$$V_{Both}(B) = \max_b \left\{ \frac{1}{k}(b\cos\theta + B) - \frac{1}{2k}\left((b\cos\theta + B)^2 + b^2\sin^2\theta\right) \right\}$$

So the optimal incentive contract must be such that $b^* = (1 - B)\cos\theta$, and will generate surplus:

$$V_{Both}(B) = \frac{1}{2k} - \frac{1}{2k}(1 - B)^2 \sin^2\theta - \bar{u}$$

Since we know the optimal formal incentive strength $b^*(B)$ as a function of the relational incentive strength B we can express total surplus as a function of B, and look for the optimal relational contract. The first-best can be attained by setting $B = 1$, $b = 0$. This is feasible if the principal prefers to pay the bonus rather than reverting to the optimal spot contract (or terminating the

3

Figure 2: Relational and Formal Contracts
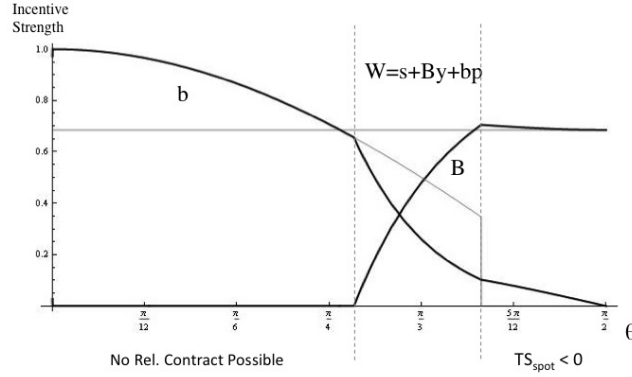
relationship entirely, if the spot contract performs very poorly). The first-best can be reached if:

$$V_{FB} - \max\{0, V_{spot}\} \geq r \quad \Rightarrow \quad \frac{1}{2k} - \bar{u} - \max\{0, \frac{1}{2k}\cos^2\theta - \bar{u}\} \geq r$$

Otherwise, we can find the second-best relational contract by solving for the largest bonus B that solves:

$$V_{Both}(b, B) - \max\{0, V_{Spot}\} \geq rB$$

which implies that B solves:

$$\frac{1}{2k} - \frac{1}{2k}(1-B)^2 \sin^2\theta - \bar{u} - \max\{0, \frac{1}{2k}\cos^2\theta - \bar{u}\} \geq rB$$

This equation is quadratic, therefore some form of relational contract can be sustained whenever this expression has a real, positive root. We can use this condition to find the cutoff value of r for which some form of relational contract can be sustained. If no such contract is possible, then only the formal contract based on p will be used.

Figure 2 shows the outcome of this combined contract. In this case the relational bonus B does depend on θ, since θ affects the termination outcome. For small θ, where the formal contract performs well, it is not possible to sustain a relational contract. However as the surplus generated by the formal contract declines, it is possible to introduce a relational bonus (whilst maintaining some formal incentives). Moreover, supplementing the relational bonus with some formal incentives can generate enough surplus to achieve a bonus B greater than that which may be reached through a relational contract alone.

4

2 Introducing an asset: employment vs outsourcing

This section introduces an asset that may be owned by either the principal or the agent, and the value of which is affected by the actions the agent takes (e.g. efforts made to maintain the value of the asset). Like output, the value of the asset is non-contractible.

- Output: $\Pr[y = 1] = a_1$

- Asset Value: $\Pr[v = 1] = a_2$

- Performance Measure: $\Pr[p = 1] = a_1 \cos \theta + a_2 \sin \theta + \varepsilon a_3$

- Cost of effort: $c(a_1, a_2, a_3) = \frac{k}{2}(a_1^2 + a_2^2 + a_3^2)$

- Outside Options: $\bar{\pi} = \bar{u} = 0$ (for simplicity, let the agent's outside option be set to zero)

NB: for simplicity, the notation $y = f.a$, $v = h.a$, $p = g.a$ may be used, where $a = (a_1, a_2, a_3)$, $f = (1, 0, 0)$, $h = (0, 1, 0)$, $g = (\cos \theta, \sin \theta, \varepsilon)$.

2.1 First Best

The first-best choice of effort maximises:

$$V_{FB} = \max_a \left\{ a_1 + a_2 - \frac{k}{2}(a_1^2 + a_2^2 + a_3^2) \right\}$$

which implies that $a^* = \left(\frac{1}{k}, \frac{1}{k}, 0 \right)$. This generates surplus $V_{FB} = \frac{1}{k}$

2.2 Optimal Static Contract

Consider formal contracts of the form $w = s + bp$. Start by finding the optimal contract under employment (P owns the asset) and under outsourcing (A owns the asset). Comparing the surplus generated by each contract yields the optimal allocation of the asset under the static (spot) contract.

2.2.1 Spot Outsourcing

If the agent owns the asset, then effort is chosen to maximise:

$$U_A(b) = \max_a \{a_2 + s + b(a_1 \cos \theta + a_2 \sin \theta + \varepsilon a_3) - \frac{k}{2}(a_1^2 + a_2^2 + a_3^2)\}$$

so $a = \frac{bg + h}{k} = \frac{1}{k}(b \cos \theta, b \sin \theta + 1, b\varepsilon)$.

The principal then chooses the optimal formal incentive b to maximise:

$$V_{Spot}^O = \max_b \{a_1 + a_2 - \frac{k}{2}(a_1^2 + a_2^2 + a_3^2)\} = \max_b \{\frac{(bg + h)}{k}.(1, 1, 0) - \frac{1}{2k}(bg + h)^2\}$$

so $b^* = \frac{f.g}{||g||^2} = \frac{\cos \theta}{1 + \varepsilon^2}$ and total surplus is

$$V_{Spot}^O = \frac{1}{2k} \left[\frac{\cos^2 \theta}{1 + \varepsilon^2} + 1 \right]$$

5

2.2.2 Spot Employment

In this case the principal owns the asset so the agent choose effort to maximise:

$$U_A(b) = \max_a \{s + b(a_1 \cos\theta + a_2 \sin\theta + \varepsilon a_3) - \frac{k}{2}(a_1^2 + a_2^2 + a_3^2)\}$$

so $a = \frac{b}{k}(\cos\theta, \sin\theta, \varepsilon) = \frac{b}{k}g$.

The principal then chooses b to maximise total surplus:

$$V_{Spot}^E = \max_b \{a_1 + a_2 - \frac{k}{2}(a_1^2 + a_2^2 + a_3^2)\} = \max_b \{\frac{b}{k}(1,1,0).g - \frac{1}{2k}b^2||g||^2\}$$

therefore $b^* = \frac{(1,1,0).g}{||g||^2} = \frac{\cos\theta + \sin\theta}{1+\varepsilon^2}$, and the resulting total surplus is:

$$V_{Spot}^E = \frac{(\cos\theta + \sin\theta)^2}{2k(1+\varepsilon^2)}$$

2.2.3 Optimal Allocation of the Asset

We have found the optimal static contract under both employment and outsourcing, and the resulting surplus. Comparing the surplus generated under each allocation determines the optimal ownership of the asset. For example, for small θ the performance measure is closely aligned with output, but not with the value of the asset. It therefore provides good incentives for a_1 but poor incentives for a_2, so giving ownership of the asset to the agent will balance incentives, and for small θ and small ε, U_{Spot}^O approaches the first best. In contrast, as $\theta \to \frac{\pi}{4}$ the performance measure balances incentives for a_1 and a_2, so it is undesirable to provide further incentives for a_2 by giving the asset to the agent. Therefore employment (integration) is preferred for larger θ.

We can solve for the cutoff θ^{SP} at which it is optimal to switch from outsourcing to employment: θ^{SP} solves

$$\frac{1}{2k}\left[\frac{\cos^2\theta^{SP}}{1+\varepsilon^2} + 1\right] = \frac{(\cos\theta^{SP} + \sin\theta^{SP})^2}{2k(1+\varepsilon^2)}$$

6

2.3 Relational Incentive Systems

Now consider the optimal relational incentive contract. Fixing the governance structure (either employment or outsourcing), for each θ we will find the optimal relational contract, and the values of r for which that contract is feasible. Note that the spot governance structure (which determines the value of the outside option should the relationship terminate) is not fixed; that is, terminating the relationship may require a change in ownership of the asset to get to the optimal spot contract and governance structure.

Having found the optimal contract under each governance structure, we can compare the surplus generated to find the optimal relational governance structure $g^*(\theta)$. Again, this may not be the same governance structure as would be chosen under the static contract.

We will allow the optimal relational contract to include bonus payments based on both y and v, so we consider contracts of the form $w = s + bp + By + \beta v$. It can be shown that this contract cannot be improved upon by adding interaction terms.

The relational contract can be implemented if:

$$-max\left\{s + bp + By + \beta v\right\} + \frac{1}{r}V_{Rel}^P \geq \frac{1}{r}V_{Spot}^P - \bar{\pi}, \text{ and}$$

$$min\left\{s + bp + By + \beta v\right\} + \frac{1}{r}V_{Rel}^A \geq \frac{1}{r}V_{Spot}^A - \bar{u}.$$

$$\Rightarrow \frac{1}{r}\left(V_{Rel} - V_{Spot}\right) \geq max\left\{0, B, \beta, B + \beta\right\} - min\left\{0, B, \beta, B + \beta\right\} = |B| + |\beta|.$$

2.3.1 Relational Outsourcing

Agent's Problem: First consider the case in which the agent owns the asset, so is the residual claimant of v. The worker maximises:

$$U^O = s + bp + By + \beta v + v - \frac{1}{2}ka^2 = s + bg.a + Bf.a + (1 + \beta)h.a - \frac{1}{2}ka^2$$

$$\Rightarrow a = \frac{bg + Bf + (1 + \beta)h}{k}$$

In this case the total surplus is:

$$V_{Rel}^O = (f + h)\left(\frac{(1+\beta)h + Bf + bg}{k}\right) - \frac{1}{2k}\left((1 + \beta)h + Bf + bg\right)^2$$

Principal's Problem: Maximising over b in order to find the optimal level of formal incentives as a function of the relational bonus payments, we obtain

$$b(B, \beta) = \frac{(1 - B)\cos\theta - \beta\sin\theta}{1 + \varepsilon^2}$$

The total surplus becomes

$$V_{Rel}^O = \frac{1}{k}\left[(1, 1, 0)(B + b\cos\theta, 1 + \beta + b\sin\theta, b\varepsilon) - \frac{1}{2}(B + b\cos\theta, 1 + \beta + b\sin\theta, b\varepsilon)^2\right]$$

$$V_{Rel}^O = U^{FB} - \frac{1}{2k}\left[(1 - B)^2 + \beta^2 - \frac{((1 - B)\cos\theta - \beta\sin\theta)^2}{1 + \varepsilon^2}\right] \quad (1)$$

7

First-Best From this expression we can see that the first best can only be reached by setting $B = 1$, $\beta = 0$. This will be possible if

$$V_{FB} - V_{Spot} \geq r$$

We can then find the cutoff $r^{FB}(\theta)$ at which it is possible to implement the first-best relational contract. This is shown in figure 3.

Second-Best If the first best cannot be reached then the reneging constraint will bind and the optimal values of B, β will solve:

$$max_{B,\beta} \left\{ U^{FB} - \frac{1}{2k} \left[(1-B)^2 + \beta^2 - \frac{((1-B)\cos\theta - \beta \sin\theta)^2}{1 + \varepsilon^2} \right] \right\}$$

subject to

$$V^{FB} - \frac{1}{2k} \left[(1-B)^2 + \beta^2 - \frac{((1-B)\cos\theta - \beta \sin\theta)^2}{1 + \varepsilon^2} \right] - V_{Spot} = r\left(|B| + |\beta| \right)$$

Assume that $B > 0$ and $\beta < 0$ (this turns out to be the only relevant case), so that the reneging temptation becomes $(B - \beta)$. Solving the maximisation problem gives the following relationship between B and β:

$$\frac{\beta}{1 - B} = \frac{\sin\theta\cos\theta - \sin^2\theta - \varepsilon^2}{\sin\theta\cos\theta - \cos^2\theta - \varepsilon^2} \tag{2}$$

The reneging constraint will bind, so we can solve for the optimal B in terms of r and θ, and hence obtain β and B. These values of B and β will solve the problem for all values of r, so in this case checking feasibility requires solving for those values of r, θ for which it is in fact the case that $B > 0$ and $\beta < 0$, as assumed. (If other assumptions on the signs of B, β are made, then there are no values of r for which these assumptions prove to be true.) This gives the cutoff for when a relational contract on v and y is possible under outsourcing.(NB The algebra here becomes intractable, but the solutions and the boundary for r can be found using Mathematica.) The boundary at which a second-best relational contract based on y and v is feasible is also depicted in figure 3.

Corner Solutions The other possibility to be checked is whether there are corner solutions to this problem - i.e., can we increase the set of values $r(\theta)$ under which a relational contract is possible by considering contracts based solely on output, or solely on the value of the asset. It turns out that it may still be possible to use a relational contract in which $B = 0$ and $\beta < 0$ when other relational contracts cannot be supported. Under outsourcing, the agent already has full strength incentives to maintain the asset, but this relational contract strengthens the formal incentives on p, whilst using the bonus payment β to offset the incentive to distort effort towards a_2.
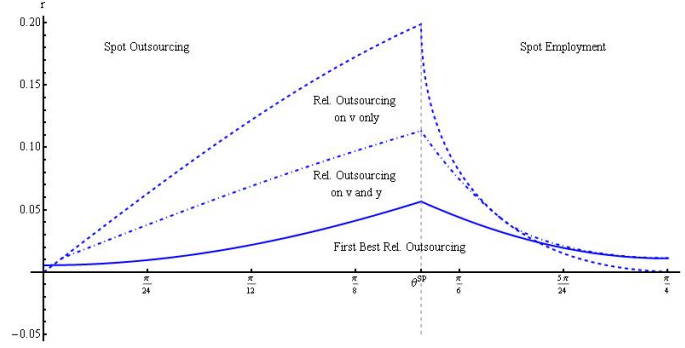
Setting $B = 0$, the total surplus becomes:

$$V_{Rel}^O = V^{FB} - \frac{1}{2k} \left[\beta^2 + 1 - \frac{(\cos\theta - \beta\sin\theta)^2}{1 + \varepsilon^2} \right] \tag{3}$$

The first best cannot be reached if there can only be relational contracting on v. However, the relational contract will still be able to improve on the static optimum as long as $V_O^{Rel} - V_{Spot} > r\,|\beta|$. Solving for the optimal value of β,

$$\beta^* = -\frac{\sin\theta\cos\theta}{\cos^2\theta + \varepsilon^2}$$

Figure 3: Values of r for which a relational contract is feasible under outsourcing

This solution will be feasible as long as:

$$\frac{1}{k} - \frac{1}{2k}\left[\left(\frac{sin\theta cos\theta}{cos^2 + \varepsilon^2}\right)^2 + 1 - \frac{\left(cos\theta + \left(\frac{sin\theta cos\theta}{cos^2\theta + \varepsilon^2}\right)sin\theta\right)^2}{1 + \varepsilon^2}\right] - U^S \geq r\left(\frac{sin\theta cos\theta}{cos^2\theta + \varepsilon^2}\right)$$

As long as this constraint is not binding, the principal can set $\beta = \beta^*$. Otherwise, if the constraint is binding, we can solve for the largest bonus payment β that satisfies this constraint. As long as this equation has a real solution, a second-best relational contract based on v is possible; the values $r(\theta)$ for which a relational contract of this form is possible are shown in figure 3.

2.3.2 Relational Employment

Agent's Problem: The principal now owns the asset, so the worker maximises:

$$U^E = s + bp + By + \beta v - \frac{1}{2}ka^2 = s + bg.a + Bf.a + (1 + \beta)h.a - \frac{1}{2}ka^2$$

$$\Rightarrow a = \frac{bg + Bf + \beta h}{k}$$

In this case the total surplus is:

$$V_{Rel}^E = (f + h)\left(\frac{\beta h + Bf + bg}{k}\right) - \frac{1}{2k}(\beta h + Bf + bg)^2$$

Principal's Problem: Maximising over b, as before

$$b = \frac{(1 - B)\cos\theta + (1 - \beta)\sin\theta}{1 + \varepsilon^2}$$

In this case the total surplus becomes

$$V_{Rel}^E = \frac{1}{k}\left[(1, 1, 0)(B + b\cos\theta, \beta + b\sin\theta, b\varepsilon) - \frac{1}{2}(B + b\cos\theta, \beta + b\sin\theta, b\varepsilon)^2\right]$$

$$V_{Rel}^E = V^{FB} - \frac{1}{2k}\left[(1 - B)^2 + \beta^2 - \frac{((1 - B)\cos\theta + (1 - \beta)\sin\theta)^2}{1 + \varepsilon^2}\right] \tag{4}$$

9

First-Best: The first best can only be reached by setting $B = 1$, $\beta = 1$. This will be possible if

$$V_{FB} - V_{Spot} \geq 2r$$

The range of values $r(\theta)$ for which the first-best is attainable under an employment contract is shown in figure 4.

Second-Best: If the first best cannot be reached then the optimal values of B, β will solve:

$$max\left[V^{FB} - \frac{1}{2k}\left[(1-B)^2 + \beta^2 - \frac{((1-B)\cos\theta + (1-\beta)\sin\theta)^2}{1+\varepsilon^2}\right]\right]$$

subject to

$$V^{FB} - \frac{1}{2k}\left[(1-B)^2 + \beta^2 - \frac{((1-B)\cos\theta + (1-\beta)\sin\theta)^2}{1+\varepsilon^2}\right] - U^S = r\left(|B| + |\beta|\right)$$

This time assume that $B > 0$ and $\beta > 0$, (again, this proves to be the only relevant case), so that the maximum reneging temptation will be $(B + \beta)$. The optimal values of B and β must satisfy.

$$\frac{1-\beta}{1-B} = \frac{\sin\theta\cos\theta + \sin^2\theta + \varepsilon^2}{\sin\theta\cos\theta + \cos^2\theta + \varepsilon^2} \qquad (5)$$

Inserting this into the reneging constraint will yield the optimal values of B and β, and as above, this contract will be feasible as long as our assumptions that $B, \beta > 0$ hold. We can then derive the cutoff values $r(\theta)$ for which the contract is possible.

Corner Solutions: Again, we can check for corner solutions in which there is a relational contract on either output or the asset only. As in the outsourcing case, a bonus based on v only can increase the range of values $r(\theta)$ for which relational contracting is possible.

With $B = 0$, the total surplus becomes:

$$V_{Rel}^E = V^{FB} - \frac{1}{2k}\left[(1-\beta)^2 + 1 - \frac{(\cos\theta + (1-\beta)\sin\theta)^2}{1+\varepsilon^2}\right] \qquad (6)$$

The first-best cannot be reached if there can only be relational contracting on v. However, the relational contract will still be able to improve on the static optimum as long as $V_{Rel}^E - V^{Spot} > r\left(|\beta|\right)$. Solving for the optimal value of β,

$$\beta^* = 1 - \frac{\sin\theta\cos\theta}{\cos^2\theta + \varepsilon^2}$$

This solution will be feasible as long as:

$$\frac{1}{k} - \frac{1}{2k}\left[\left(\frac{\sin\theta\cos\theta}{\cos^2 + \varepsilon^2}\right)^2 + 1 - \frac{\left(\cos\theta + \left(\frac{\sin\theta\cos\theta}{\cos^2\theta + \varepsilon^2}\right)\sin\theta\right)^2}{1+\varepsilon^2}\right] - V_{Spot} > r\left(1 - \frac{\sin\theta\cos\theta}{\cos^2\theta + \varepsilon^2}\right)$$

If this constraint binds, then as above we can solve for the largest bonus payment B that is credible. A second-best relational contract of this form will be feasible whenever this equation has a real solution. The resulting range $r(\theta)$ for which this is the case is shown in figure 4.

Figure 4: Values of r for which a relational contract is feasible under emploment
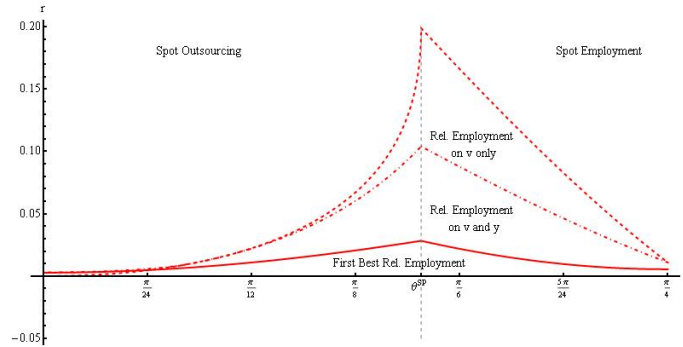

2.3.3 Horserace

Having found the set of feasible relational contracts under both outsourcing and integration, for all θ, it remains to establish which governance structure is optimal. For some values of θ, this answer is clear: relational contracts are only sustainable under one of the possible governance structures. However for intermediate values of θ it may be possible to maintain a relational contract under either governance structure (NB - not the same relational contract). In this case comparing the surplus generated by the optimal relational contract under each governance structure allows us to establish the optimal allocation of the asset.

Fix $r = 0.07$. From above, we know which contracts are possible under both employment and outsourcing for all θ, and moreover we know the form of such contracts.

Figure 5: Feasibility of relational contracts for $r = 0.07$
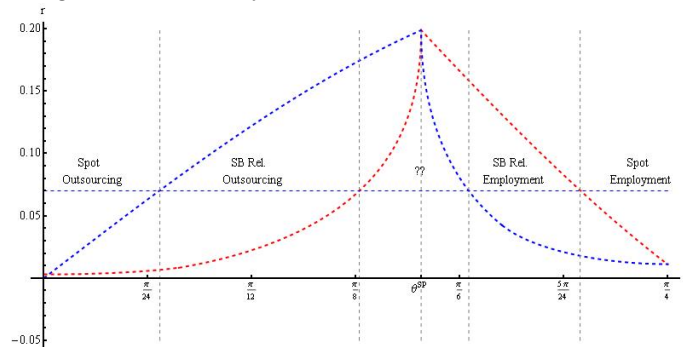
As figure 5 shows, for intermediate θ relational contracts are feasible under both systems of governance, so it is unclear which governance structure should be chosen. In this case we can compare the surplus generated by the optimal contract in each case, shown in figure 6:

11

Figure 6: Surplus generated by the optimal contract under employment, outsourcing
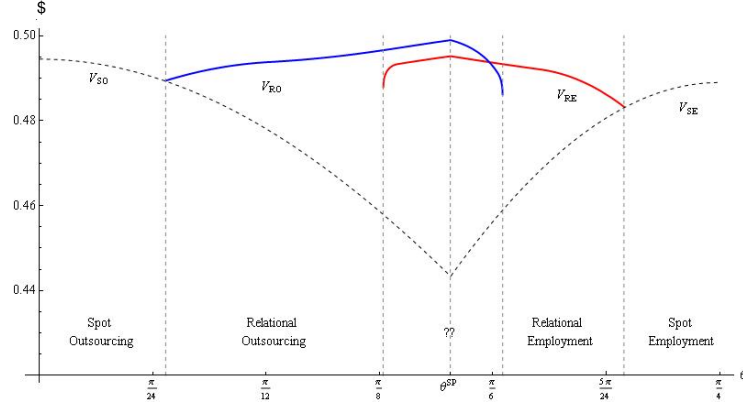
Solving for the value of θ at which the principal and agent are indifferent between the two possible governance structures, we can determine the optimal allocation of the asset $g(\theta)$. Note that this value of θ need not coincide with θ^{SP}, so the optimal governance structure may differ between the purely formal and the relational contract. This also implies that should the relationship be terminated, the asset could potentially change hands when the parties revert to the formal contract. Figure 7 shows the optimal governance structures and the associated contracts.

Figure 7: Optimal relational contract, given θ, $r = 0.07$